International Journal of Theoretical Physics, Vol. 43, Nos. 7/8, August 2004 (© 2004)

Note on Jordan Groupoids

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This paper is a continuation of a previous article by the author (Katrnoška, F., 1995, *International Journal of Theoretical Physics*, **34**(8), 1501–1505). It contains some further results that concern the left and right Jordan groupoids. An investigation is presented of the relation between the rings with identities and their corresponding left (right) Jordan groupoids. Orthoposets give examples of Jordan groupoids, hence the results obtained may find an application in the foundation of quantum theory.

KEY WORDS: ring with identity; *ring with identity; ring homomorphism; idempotents of a *ring; projectors of a *ring; category; functor.

1. INTRODUCTION

Let R be an associative ring with identity 1 and let $U(R)$ be the set of all idempotents of the ring R. Suppose that R is a *ring. An idempotent $p \in R$ is said to be a projector of R, if $p = p^*$. Let us define $P(R)$ as the set of all projectors of the *ring R. On the sets $U(R)$ resp. $P(R)$ we can define binary operations $\circ$ and $\square$ as follows:

$$p \circ q = (1 - 2q)p(1 - 2q)$$

$$p \square q = (1 - 2p)q(1 - 2p)$$

if $p, q \in U(R)$ (resp. $p, q \in P(R)$). Remember that if R is a ring with identity. 1 and if $p \in U(R)$, then the element $a = 1 - 2p$ is invertible and $a^{-1} = 1 - 2p$ (See Katrnoška, 1993). The sets $U(R)$ and $P(R)$ always contain the elements 0, 1 and certainly also $p' = 1 - p \in U(R)$ whenever $p \in U(R)$ (resp. p' whenever $p \in P(R)$). The groupoids $U(R)$ resp. $P(R)$ are in general nonassociative, noncommutative with respect to the operations $\circ$ and $\square$. Nevertheless, both groupoids are elastic, i.e., they fulfil the conditions $(p \circ q) \circ p = p \circ (q \circ p)$, resp. $(p \square q) \square p = p \square (q \square p)$, where $p, q \in U(R)$, resp. $p, q \in P(R)$. There exist mutual relations between the homomorphisms of the rings with identities and the corresponding left (right) Jordan groupoids.

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2. THE LEFT AND THE RIGHT JORDAN GROUPOIDS

We formalize now the whole situation and give at first the definitions of the left and the right Jordan groupoids. We show also some of their properties.

Definition 2.1. (Katrnoška, 1993). A nonempty set X is said to be a left Jordan groupoid if there is a binary operation $\circ : X \times X \to X$ and a unary operation $': X \to X$ (orthocomplementation on X) such that the following conditions are satisfied.

(i) $p \circ p = p, \quad p \in X,$
(ii) $(p \circ q) \circ p = p \circ (q \circ p), \quad p, q \in X,$
(iii) $(p \circ q) \circ q = p, \quad p, q \in X,$
(iv) $(p')' = p, \quad p \in X,$
(v) $(p \circ q)' = p' \circ q', \quad p, q \in X,$
(vi) $p \circ q' = p \circ q, \quad p, q \in X,$
(vii) X contains elements $0, 1 \in X$ such that $p \circ 1 = p \circ 0 = p$, $1 \circ p = 1$, $0 \circ p = 0$, and $0' = 1$, $p \in X$

Definition 2.2. A nonempty set X is said to be a right Jordan groupoid if there exists a binary operation $\square : X \times X \to X$ and a unary operation $' : X \to X$ such that the following conditions are satisfied.

(i) $p \square p = p, \quad p \in X,$
(ii) $(p \square q) \square p = p \square (q \square p), \quad p, q \in X,$
(iii) $q \square (q \square p) = p, \quad p, q \in X,$
(iv) $(p')' = p, \quad p \in X,$
(v) $p' \square q' = (q \square p)', \quad p, q \in X,$
(vi) $q' \square p = q \square p, \quad p, q \in X,$
(vii) X contains elements $0, 1 \in X$ such that, for each $p \in X$, $p \square 1 = 1$, $p \square 0 = 0$, $1 \square p = 0 \square p = p$, and $0' = 1$

Both above-introduced groupoids were called by me Jordan groupoids because then they fulfil the Jordan condition, (see Katrnoška, 1999),

$$(p \circ p) \circ (q \circ p) = ((p \circ p) \circ q) \circ p \qquad p, q \in X.$$

We exhibit now an example of a left and of a right Jordan groupoid.

Example 2.1. Let R be an associative ring with identity and let $U(R)$ be the set of all idempotents of the ring R. The operations $\circ$ and $\square$ on $U(R)$ are defined by setting

$$p \circ q = (1 - 2q)p(1 - 2q), \quad p, q \in U(R)$$

$$p \square q = (1 - 2p)q(1 - 2p), \quad p, q \in U(R)$$

As an orthocomplement p' of $p \in U(R)$ we take the element $p' = 1 - p$. Of course, $p' \in U(R)$. It is possible to show that $(U(R), \circ, 0, 1,')$ is a left Jordan groupoid and $(U(R), \square, 0, 1,')$ is a right Jordan groupoid.

The mutual relation between the left and the right Jordan groupoids $(X, \circ, 0, 1,')$ and $(X, \square, 0, 1,')$ is given as follows:

Proposition 2.1. *Let $(X, \circ, 0, 1,')$ be a left Jordan groupoid. Let us define a binary operation $\square : X \times X \to X$ by $p \square q = q \circ p$, $p, q \in X$. Then $(X, \square, 0, 1,')$ is a right Jordan groupoid.*

The proof is easy.

In the following proposition the necessary and sufficient condition for associativity of the left Jordan groupoid is given.

Proposition 2.2. *Let $(X, \circ, 0, 1,')$ be a left Jordan groupoid. Then $(X, \circ, 0, 1,')$ is associative iff $p \circ q = p$ for each $p, q \in X$.*

Proof: The condition is necessary: Let $(X, \circ, 0, 1,')$ be an associative left Jordan groupoid. According to (i) and (iii) of the Definition we have $p \circ q = p \circ (q \circ q) = (p \circ q) \circ q = p$ if $p, q \in X$. The condition is sufficient: Suppose that $p \circ q = p$ for all $p, q \in X$. Then $(p \circ q) \circ r = p \circ r = p = p \circ (q \circ r)$ if $r \in X$. Therefore $(X, \circ, 0, 1,')$ is an associative left Jordan groupoid. □

Analogously the right Jordan groupoid $(X, \circ, 0, 1,')$ is associative iff $p \square q = q$ for each $p, q \in X$.

Lemma 2.1. *Let R be an associative ring with identity and let $(U(R), \circ, 0, 1,')$ be the left Jordan groupoid of all idempotents of the ring R. Suppose that $pq = qp$ if $p, q \in U(R)$. Then it follows*

(i) *$pq \in U(R)$ if $p, q \in U(R)$*
(ii) *$p \circ q = p$ if $p, q \in U(R)$.*

Proof:

(i) Let $p, q \in U(R)$. Then $(pq)^2 = p^2 \cdot q^2 = pq$ and $pq \in U(R)$.
(ii) If the elements p, q are mutually commutative ($p, q \in U(R)$), then

$$p \circ q = (1 - 2q) \circ p \circ (1 - 2q) = p - 2pq - 2qp + 4qpq = p.$$

□

Notice that if R is an associative, commutative Boolean ring (Birkhoff, 1973) then $U(R) = R$ and $(R, \circ, 0, 1,')$ is an associative left Jordan groupoid.

Definition 2.4. Let $(X, \circ, 0, 1,')$ be a left Jordan groupoid. The set Y is said to be a left Jordan subgroupoid of $(X, \circ, 0, 1,')$ if $Y \subset G$ and if Y is closed with respect to the operation $\circ$ and orthocomplementation of X. The set Y must contain the element 1. The subgroupoid of the right Jordan groupoid is defined analogously.

Definition 2.5. (Katrnoška, 1999). Let $(X, \circ, 0, 1,')$ be a left Jordan groupoid. The center $C(X)$ of X is the set of all $p \in X$ such that $p \circ q = p$ for each $q \in X$.

In author's paper (Katrnoška, 1999) it is proved that the center $C(X)$ of the left Jordan groupoid $(X, \circ, 0, 1,')$ is an associative subgroupoid of X.

Proposition 2.3. *Every left (resp. right) Jordan groupoid with at least two elements is noncommutative and not necessarily associative.*

Proof: We carry out the proof only for the left Jordan groupoids. The case of the right Jordan groupoid is analogous. According to $p \circ 1 = p$, $1 \circ p = 1$(p is an element of the given left Jordan groupoid X). We see that for $p \neq 1$ the axiom of the commutativity is not valid. In order to show that the left Jordan groupoid need not be associative we can take for example the left Jordan groupoid $(P(R), \circ, 0, 1,')$. If p, q are the elements of $P(R)$ different from 1, then it holds $p \circ (q \circ 1) = p$, but $(p \circ q) \circ 1 = p \circ q$ if $p \neq q$. It need not hold in general $p \circ q = p$. □

We give an example of an associative left Jordan groupoid.

Example 2.6. Let us consider the set $X = \{p, p', q, q', \ldots, 0, 1\}$ (It is the modular ortholattice *MOn*, see Beran, 1995; Kalmach, 1983). If $r \in X$, then the element $r' \in X$ is an orthocomplement of r and it holds that $(r')' = r, r \in X$ and $0' = 1$. We define the binary operation $\circ$ by the following condition $p \circ q = p$ for each $p, q \in X$. It is possible to prove that $(X, \circ, 0, 1,')$ is a left Jordan groupoid. According to Proposition 2.2 this left Jordan groupoid is associative and the obvious relation $0' = 1$ is valid in X.

Proposition 2.4. *Let R be a *ring with the identity and let $P(R)$ be the set of all projectors of the *ring R. Then $(P(R), \circ, 0, 1,')$ is a left subgroupoid of $(U(R), \circ, 0, 1,')$.*

Proof: Suppose that $p, q \in P(R)$. Then

(i) $(p \circ q)^* = [(1-2q)p(1-2q)]^* = (1-2q)^* p^* (1-2q)^* = (1-2q) p(1-2q) = p \circ q$. Therefore $p \circ q \in P(R)$.
(ii) Let us take $p \in P(R)$. We have certainly $(p')^* = (1-p)^* = 1 - p^* = (p^*)'$. Therefore $(p')^* = p'$ and $p' \in P(R)$.
(iii) Since $1^* = 1$ and $0^* = 0$ then $1, 0 \in P(R)$. □

Example 2.7. (generalization of Ex. 2.6) Let $(X, \leq, 0, 1,')$ be an orthoposet [1]. If we define a binary operation $\circ : X \times X \to X$ by $p \circ q = p$, $p, q \in X$, then $(X, \circ, 0, 1,')$ is an associative left Jordan groupoid. The book (Birkhoff, 1973) deals only with the theory of orthomodular lattices. Concerning different questions of orthoposets see also Flachsmeyer (1982).

3. THE HOMOMORPHISMS OF THE LEFT (RIGHT) JORDAN GROUPOIDS

We introduce at first some further definitions.

Definition 3.8. Let $(X_1, \circ_1, 0_1, 1_1,')$ and $(X_2, \circ_2, 0_2, 1_2,^*)$ be left Jordan groupoids. The set $(X_1 \times X_2, \circ, 0, 1,^+)$ in which $(p_1, q_1) \circ (p_2, q_2) = (p_1 \circ_1 p_2, q_1 \circ_2 q_2)$, $p_1, p_2 \in X_1, q_1, q_2 \in X_2, 0 = (0_1, 0_2), 1 = (1_1, 1_2)$ and $(p, q)^+ = (p', q^*)$ if $p \in X_1, q \in X_2$ is said to be a cartesian product of the left Jordan groupoids X_1 and X_2. It is possible to prove that $(X_1 \times X_2, \circ, 0, 1,^+)$ is always a left Jordan groupoid.

Definition 3.9. Let $(X_1, \circ_1, 0_1, 1_1,')$ and $(X_2, \circ_2, 0_2, 1_2,^*)$ be left Jordan groupoids. Suppose that the mapping $h : X_1 \to X_2$ satisfies the following conditions.

(i) $h(p_1 \circ_1 p_2) = h(p_1) \circ_2 h(p_2)$, $p_1, p_2 \in X_1$
(ii) $h(p') = [h(p)]^*$, $p \in X_1$
(iii) $h(0_1) = 0_2$

Then the mapping h is called a homomorphism of X_1 into X_2. A bijective homomorphism h of $(X_1, \circ, 0, 1,')$ onto itself is said to be an automorphism.

We exhibit now an example of a homomorphism.

Example 3.10. Let $(X_1, \circ_1, 0_1, 1_1,')$ and $(X_2, \circ_2, 0_2, 1_2,^*)$ be left Jordan groupoids and let $(X_1 \times X_2, \circ, 0, 1,^+)$ be a cartesian product of the left Jordan groupoids X_1 and X_2. Consider the mapping $h : X_1 \times X_2 \to X_2$ defined by $h(p_1, p_2) = p_2$, $p_1 \in X_1$, $p_2 \in X_2$. It can be shown that h is a homomorphism of $X_1 \times X_2$ onto X_2.

Further we show an example of an automorphism.

Example 3.11. Let R be a *ring with identity. As the left Jordan groupoid we take $(U(R), \circ, 0, 1,')$ (see Example 2.3). Define the mapping $h : U(R) \to U(R)$ by the following manner: $h(p) = p^*$, $p \in U(R)$. It is possible to show that h is an automorphism of $(U(R), \circ, 0, 1,')$.

Remark 3.12. If R is an associative ring with identity 1 and if $p \in U(R)$, then the mapping $h_p : R \to R$ defined by setting $h_p(x) = (1-2p)x(1-2p)$, $x \in R$, is an automorphism of R. This automorphism fulfills the following conditions: $h_p(h_p(x)) = x (x \in R)$, and $h_p(U(R)) = U(R)$.

Some further connections concerning the automorphisms of rings and orthoposets can be found in the papers Chevalier (1993) and Pulmannová (1996).

Now we introduce the following statement (without proof): If all idempotents of R mutually commute, then each mapping h_p, $p \in U(R)$ is an automorphism of both Jordan groupoids $(U(R), \circ, 0, 1')$ and $(U(R), \Box, 0, 1')$.

Corollary 3.1. *Let R be an associative ring with identity 1 and let $p, q \in U(R)$. If h_p is the automorphism introduced in Remark 3.12, then we can also define the binary operations $\circ$ and $\Box$ as follows:*

$$p \circ q = h_q(p), \quad p \Box q = h_p(q).$$

Now we introduce a further proposition which gives certain characterization of the center $C(U(R))$ of the left Jordan groupoid $(U(R), \circ, 0, 1,')$.

Proposition 3.5. *Let $(U(R), \circ, 0, 1,')$ be a left Jordan groupoid. The center $C(U(R))$ of this groupoid may be expressed in the following from:*

$$C(U(R)) = \{p \in U(R) : (\forall q \in U(R) : h_q(p) = p)\}.$$

Proof: It is a consequence of Definiton 2.5 and Corollary 3.1. □

Remember that automorphisms of rings and of rings with identity are very useful and suitable notions in quantum theory investigations.

Remark that the statement "R is an associative ring with identity" means also that R is a unitary ring (Pareigis, 1969; Semadeni and Wiweger, 1978).

Definition 3.13. A homomorphism h of a unitary ring R_1 into a unitary ring R_2 is said to be a unitary homomorphism if the following conditions are satisfied:

(i) $h(a+b) = h(a) + h(b)$, $a, b \in R_1$
(ii) $h(a \cdot b) = h(a) \cdot h(b)$, $a, b \in R_1$
(iii) $h(1_1) = 1_2$

Proposition 3.6. *Let R_1 and R_2 be unitary rings and let $(U(R_1), \circ_1, 0_1, 1_1,')$, $(U(R_2), \circ_2, 0_2, 1_2,')$ be the respective left Jordan groupoids of all idempotents of the rings R_1 and R_2 (see Example 2.3). If $h : R_1 \to R_2$ is a unitary homomorphism of the ring R_1 into the ring R_2 then the restriction $h|U(R_1)$ of the unitary* *homomorphism h to the left Jordan groupoid $U(R_1)$ is a homomorphism of the left Jordan groupoid $U(R_1)$ into $U(R_2)$.*

The proof is easy.

Similar proposition is also valid for the right Jordan groupoids. Some connections with the contents of this paper may be found in Chevalier (1993), Katrnoška (1993), and Pták, S. Pulmanová (1991).

4. CATEGORIAL CONSEQUENCES

Using Proposition 3.6 we shall show some conclusions which are formulated using the theory of categories and functors. If $\mathcal{K}$ is a given category, then $\mathcal{K}^{\mathrm{o}}$ denotes the class all elements (objects) of the category $\mathcal{K}$.

Notation Let us denote by *Ri* the category of all unitary rings (Pareigis, 1969) and let us further denote by *LJG* (resp. *RJG*) the category of all left (right) Jordan groupoids.

We can define a functor F from the category *Ri* to the category *LJG*. The object transformation $F : (Ri)^{\mathrm{o}} \to (LJG)^{\mathrm{o}}$ assigns to each unitary ring R (i.e., $R \in (Ri)^{\mathrm{o}}$ a left Jordan groupoid $U(R)$ (i.e., $U(R) \in (LJG)^{\mathrm{o}}$ while the morphism transformation of the functor F assigns to each unitary homomorphism $h : R_1 \to R_2$, $R_1, R_2 \in (Ri)^{\mathrm{o}}$ a homorphism $h_{|U(R_1)} : U(R_1) \to U(R_2)$.

The following proposition holds:

Proposition 4.7. *A functor $F : Ri \to LJG$ is a covariant functor.*

The proof is easy.

ACKNOWLEDGMENT

I thank L. Beran, J. Flachsmeyer, and J. Hamhalter, and especially M. Navara for suggestions and discussions.

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